UDC 539.371

Candidate of physicomathematical sciences S. A. Levchuk

National University, Zaporizhzhia

APPLYING THE APPARATUS OF GREEN MATRIXES AND MATRIX ALGEBRA TO THE ISSUE OF STATIC DEFORMATION OF CIRCULAR PLATES WITH DISCRETELY VARIABLE THICKNESS

The article devoted for modelling of static deformation of the circular plates with discrete-variable thickness. The matrix of Green type and algebra of matrix had been used what allow had been constructed compact computing algorithm for solution of consider problem. Method of calculation, which propose, had been generalised on the case n section in the circular plate.

Key words: *circular plate with discrete-variable thickness, boundary-compound problem, compound construction, matrix of Green type, algebra of matrix.*

The article is devoted to modelling static deformation of circular plates with discretely variable thickness. Such approach is not new. It was previously developed in several works [3–5, 7, 8]. But in this paper the application of the Green function apparatus and matrix algebra allowed to create a compact computational algorithm of solving the considered task in conditions of almost unrestricted quantity of sections in compound body which was used for modelling. The approach mentioned was fully realized in the work [6], but for a ring-shaped plate with variable thickness.

Materials and methods of study

Let's have a look at a circular plate with discretely variable thickness (fig. 1).

The calculation scheme for such model may be defined in the following way. The axisymmetric regular bending $W = W(r)$ should fit the equation [1]:

$$\Delta\Delta\, W = F\,, \tag{1}$$

where $F = q / D$, $q = q(r)$ denotes the intensity of external regular load, D denotes cylindrical rigidity of material, Δ denotes the axisymmetric Laplace operator:

$$\Delta = \frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr}\,. \tag{2}$$

Taking into account the axial symmetry, the solution should be determined only in the line of radius of the plate which can be viewed as a compound object that consists of a circular plate of the radius R_1 and a certain quantity of ring-shaped plates for which the following is true: $R_i \le r \le R_{i+1}$ $(i = 1,2,\dots,n-1)$.

The fundamental system of solutions for corresponding homogeneous equation may be such function systems [1]:

1) for a circular plate ($0 \le r \le R_1$):

$$W^{(1)} = 1\,,\quad W^{(2)} = r^2\,; \tag{3}$$

2) for the rings ($R_i \le r \le R_{i+1}$) $(i = 1,2,\dots,n-1)$:

$$W^{(1)} = 1\,,\; W^{(2)} = \ln(r),\; W^{(3)} = r^2\,,\; W^{(4)} = r^2\ln(r)\,. \tag{4}$$

Thus, the general solutions for equation (1) can be written down as follows:

$$W_1(r) = C_1(r) + C_2(r)r^2,$$

$$W_k = C_j(r) + C_{j+1}(r)\ln(r) + C_{j+2}(r)r^2 + \\ + C_{j+3}(r)r^2\ln(r), \tag{5}$$

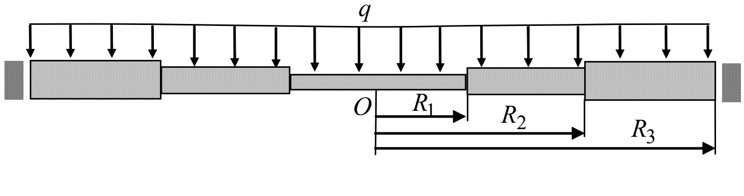

Fig. 1. Axial section of a circular plate with discretely variable thickness

where $W_1(r)$ is a regular bending of a circular plate with the radius R_1, $W_k(r)$ is a regular bending of the *k*-th ring-shaped section of the plate, $(k = 2,3,...,n)$, and the index j increases by four unities when k increases by one, whereas $k = 2$ corresponds to $j = 3$.

If we continue the solution by the method of variation of constants, then having defined C_j and having substituted their formulas in (5) it is not difficult to get dependences (accurate within constants of integration):

$$W_1(r) = \overline{C}_1 + \overline{C}_2 r^2 + \int_0^r F_1(\xi)\frac{\xi}{4}\left\{\left(\xi^2 + r^2\right)\ln\frac{r}{\xi} + \xi^2 - r^2\right\}d\xi;$$

$$W_k(r) = \overline{C}_j + \overline{C}_{j+1}\ln(r) + \overline{C}_{j+2}r^2 + \overline{C}_{j+3}r^2\ln(r) + \int_0^r F_k(\xi)\frac{\xi}{4}\left\{\left(\xi^2 + r^2\right)\ln\frac{r}{\xi} + \xi^2 - r^2\right\}d\xi. \quad (6)$$

To determine the constants $\overline{C_j}$ $(j = 1,2,...,4n-2)$ in the formulas (6) one should resort to edge conditions, for example, the conditions of rigid clamp of the edge of a compound plate:

$$W_n\big|_{r=R_n} = 0; \quad \left.\frac{dW_n}{dr}\right|_{r=R_n} = 0, \quad (7)$$

and the conditions of joining the elements of a compound construction [3]:

$$W_i\big|_{r=R_i} = W_{i+1}\big|_{r=R_i}; \quad \left.\frac{dW_i}{dr}\right|_{r=R_i} = \left.\frac{dW_{i+1}}{dr}\right|_{r=R_i};$$

$$M_i\big|_{r=R_i} = M_{i+1}\big|_{r=R_i}; \; Q_i\big|_{r=R_i} = Q_{i+1}\big|_{r=R_i}, \quad (8)$$

where $i = 1,2,...,n-1$, while M_i and Q_i denote the bending moment and the transverse force correspondingly (hereinafter lower indexes denote the numbers of sections in the compound construction), for which the following formulas are true [1]:

$$M_i(r) = -\frac{E_i h_i^3}{12\left(1-\sigma_i^2\right)}\left\{\frac{d^2W_i}{dr^2} + \frac{\sigma_i}{r}\frac{dW_i}{dr}\right\};$$

$$Q_i(r) = -\frac{E_i h_i^3}{12\left(1-\sigma_i^2\right)}\frac{d}{dr}\Delta W_i, \quad (9)$$

where h_i denotes the thicknesses of sections, σ_i denotes Poisson ratio, E_i denotes Young modulus.

After substituting (6), (9) in (7), (8) we will get a $4n-2$ system of linear algebraic equations with respect to the unknown $\overline{C_j}$ with a coefficient matrix for the unknown $A = \{a_{ij}\}_{i,j=1}^{4n-2}$.

Theoretical results and their analysis

Having solved this system and substituting the obtained formulas for $\overline{C_j}$ $(j = 1,2,...,4n-2)$ into (6) we will have:

$$W_k(r) = \sum_{l=1}^{n}\int_0^{R_l} G_l(r,\xi)\overline{F_l}(\xi)d\xi, \quad (10)$$

where $\overline{F}_l(\xi) = \begin{pmatrix} F_l(\xi) & F_{l+1}(\xi)\end{pmatrix}^T$, $\overline{F}_n(\xi) = F_n(\xi)$,

$G_l(r,\xi) = \begin{pmatrix} G_{l1}(r,\xi) & G_{l2}(r,\xi)\end{pmatrix}$, $l = 1,2,..,n-1$;

$k = 1,2,..,n$; $G_l(r,\xi)$ – are green matrixes created for this task.

If by $\overline{A} = \{\overline{a}_{ij}\}_{i,j=1}^{4n-2}$ we will label the matrix which is inverse to the matrix $A = \{a_{ij}\}_{i,j=1}^{4n-2}$, that was mentioned above and introduce the following notations:

$$t_{l1}^j(\xi) = -\overline{a}_{ji}\frac{\xi}{4}\left\{\left(\xi^2 + R_l^2\right)\ln\frac{R_l}{\xi} + \xi^2 - R_l^2\right\} - \overline{a}_{ji+1}\frac{\xi}{4}\left\{2R_l\ln\frac{R_l}{\xi} + \frac{\xi^2}{R_l} - R_l\right\} + \overline{a}_{ji+2}D_l\frac{\xi}{4}\left\{2(1+\sigma_l)\ln\frac{R_l}{\xi} + (\sigma_l - 1)\frac{\xi^2}{R_l^2} + 1 - \sigma_l\right\} + \overline{a}_{ji+3}D_l\frac{\xi}{R_l};$$

$$t_{l2}^j(\xi) = \overline{a}_{ji}\frac{\xi}{4}\left\{\left(\xi^2 + R_l^2\right)\ln\frac{R_l}{\xi} + \xi^2 - R_l^2\right\} + \overline{a}_{ji+1}\frac{\xi}{4}\left\{2R_l\ln\frac{R_l}{\xi} + \frac{\xi^2}{R_l} - R_l\right\} - \overline{a}_{ji+2}D_{l+1}\frac{\xi}{4}\left\{2(1+\sigma_{l+1})\ln\frac{R_l}{\xi} + (\sigma_{l+1} - 1)\frac{\xi^2}{R_l^2} + 1 - \sigma_{l+1}\right\} - \overline{a}_{ji+3}D_{l+1}\frac{\xi}{R_l};$$

$$t_n^j(\xi) = -\overline{a}_{j4n-3}\frac{\xi}{4}\left\{\left(\xi^2 + R_n^2\right)\ln\frac{R_n}{\xi} + \xi^2 - R_n^2\right\} - \overline{a}_{j4n-2}\frac{\xi}{4}\left\{2R_n\ln\frac{R_n}{\xi} + \frac{\xi^2}{R_n} - R_n\right\}, \quad (11)$$

where $j = 1,2,..,4n-2$; $l = 1,2,..,n-1$, and the index i increases by four unities when l increases by one, whereas $l = 1$ corresponds to $j = 1$, then the components of the created Green matrixes $G_l(r,\xi)$ will take the following form:

for $k = 1$:

$$G_{l1}(r,\xi) = \begin{cases} t_{l1}^1(\xi) + t_{l1}^2(\xi) r^2, & for\ l \neq 1; \\ t_{l1}^1(\xi) + t_{l1}^2(\xi) r^2 + I_1(r,\xi), \\ for\ l = 1, \quad I_1(r,\xi) = 0 & for\ \xi > r; \end{cases}$$

$$G_{l1}(r,\xi) = t_{l2}^1(\xi) + t_{l2}^2(\xi) r^2,$$

for $k \neq 1$:

$$G_{l1}(r,\xi) = \begin{cases} t_{l1}^j(\xi) + t_{l1}^{j+1}(\xi)\ln(r) + t_{l1}^{j+2}(\xi) r^2 + t_{l1}^{j+3}(\xi) r^2 \ln(r), \\ for\ l \neq k; \\ t_{l1}^j(\xi) + t_{l1}^{j+1}(\xi)\ln(r) + t_{l1}^{j+2}(\xi) r^2 + t_{l1}^{j+3}(\xi) r^2 \ln(r) + I_l(r,\xi), \\ for\ l = k, \\ I_l(r,\xi) = 0 \quad for\ \ \xi > r; \end{cases}$$

$$G_{l2}(r,\xi) = t_{l2}^j(\xi) + t_{l2}^{j+1}(\xi)\ln(r) + t_{l2}^{j+2}(\xi) r^2 + t_{l2}^{j+3}(\xi) r^2 \ln(r);$$

$$G_n(r,\xi) = \begin{cases} t_n^{4n-5}(\xi) + t_n^{4n-4}(\xi)\ln(r) + t_n^{4n-3}(\xi) r^2 + t_n^{4n-2}(\xi) r^2 \ln(r), \\ for\ l \neq k; \\ t_n^{4n-5}(\xi) + t_n^{4n-4}(\xi)\ln(r) + t_n^{4n-3}(\xi) r^2 + t_n^{4n-2}(\xi) r^2 \ln(r) + I_n(r,\xi), \\ for\ l = k, \\ I_n(r,\xi) = 0 \quad for\ \ \xi > r, \end{cases} \qquad (12)$$

where $l = 1,2,..,n-1$; $k = 2,3,..,n$, and the index j increases by four unities when k increases by one, whereas $k = 2$ corresponds to $j = 3$,

$$I_k(r,\xi) = \frac{\xi}{4}\left\{\left(\xi^2 + r^2\right)\ln\frac{r}{\xi} + \xi^2 - r^2\right\}.$$

It is necessary to remark that in the process of solving this task there emerge certain peculiarities in the form of improper integrals of the function which is discontinuous on the left endpoint of the interval of integration:

$$\int_0^r F(\xi)\xi^3 \ln\frac{r}{\xi} d\xi; \qquad \int_0^r F(\xi)\xi \ln\frac{r}{\xi} d\xi.$$

With the help of the theory of limits it is not difficult to show that these integrals will be convergent (in case the function $F(\xi)$ is a constant or is of power nature).

Thereby, the function (10) with components (11), (12) is the solution for the investigated task (1), (7), (8). The obtained results agree with the known [2], which were got by means of complex variable theory.

We should note that in the process of finding the numerical value of the inversed matrix A^{-1}, the element of which are necessary for the construction of corresponding Green matrixes, one will have to solve $4n$ of systems each of which consists of $4n$ algebraic equation with $4n$ unknowns, where n denotes the quantity of sections in the compound object.

While solving these systems by means of one of exact methods (for example, with the help of Gaussian elimination) we often come across calculating problems, because if n is sufficiently great the inaccuracy in calculating of the unknown becomes unsatisfactory. The application of iteration method of solving the systems of algebraic equations in the cases which are investigated is extremely difficult because there is a necessity of preliminary preparing the coefficient matrix with unknowns in conditions of huge size of these matrixes.

That's why we should pay attention to the fact that the obtained matrixes have a so-called strip structure, i. e. they contain a big amount of null elements (quasidiagonal matrixes). It is common knowledge that during the process of a system of equations with quasidiagonal matrix the number of arithmetic operations and the memory capacity of a computer may be considerably lessened which boosts the accuracy of calculations.

The calculating scheme for finding the inverse matrix A^{-1}, with the application of the approach mentioned above can be the following.

On the basis of the well-known matrix equality

$$A^{-1}A = E,$$

where $A^{-1} = \{\bar{a}_{ij}\}_{i,j=1}^{4n}$ is a matrix which is inverse with respect to the given matrix $A = \{\alpha_{ij}\}_{i,j=1}^{4n}$, E denotes a unity matrix, we see that to find the unknown elements of the inverse matrix A^{-1} we should solve $4n$ of systems of linear algebraic equations of the following form:

$$(\bar{a}_{i1} \quad \bar{a}_{i2} \quad \ldots \quad \bar{a}_{i4n})A =$$
$$= (0 \quad 0 \quad 1_i \quad 0 \quad \ldots \quad 0),$$

where i denotes the number of the inverse matrix row $(i = 1,2,...,4n)$, 1_i means that the unity is the i-th component of the vector of absolute terms.

If there are three sections in a compound object $(n = 3)$ in the form of matrix the system mentioned above will be the following (for each i):

If there are three sections in a compound object $(n = 3)$ in the form of matrix the system mentioned above will be the following (for each i):

$$A^{11}C^1 + A^{12}C^2 = F^1,$$

$$A^{22}C^2 + A^{23}C^3 = F^2,$$

$$A^{33}C^3 = F^3. \qquad (13)$$

Then we can treat the system (13) for determining the vectors of the unknown C^i $(i = 1,2,3)$ in the following way.

From the first and the second equations of the system (13) we will find, applying the rules of matrix algebra, C^2 and C^3, thus:

$$C^2 = \left(A^{12}\right)^{-1}\left(F^1 - A^{11}C^1\right),$$

$$C^3 = \left(A^{23}\right)^{-1}\left(F^2 - A^{22}C^2\right),$$

$$A^{33}C^3 = F^3.$$

Substituting C^1 found in (16) into (15), we will define C^3 and C^2 so, in that way we will finish the solution of the task.

In general form, if we examine a compound ring-shaped slate which consists of n sections, we will have.

The system for defining the elements of an inverse matrix will acquire the form similar to (13):

$$A^{11}C^1 + A^{12}C^2 = F^1,$$

$$A^{22}C^2 + A^{23}C^3 = F^2,$$

$$\ldots\ldots\ldots\ldots\ldots\ldots$$

$$A^{n-1n-1}C^{n-1} + A^{n-1n}C^n = F^{n-1},$$

$$A^{nn}C^n = F^n. \qquad (17)$$

The solution system of linear algebraic equations for defining the unknown components of vectors C^1 will gain the following form:

$$(-1)^{n+1}A^{nn}(A^{n-1n})^{-1}A^{n-1n-1}\left(A^{n-2n-1}\right)^{-1}\ldots A^{22}\left(A^{12}\right)^{-1}A^{11}C^1$$
$$= F^n - A^{nn}(A^{n-1n})^{-1}F^{n-1} +$$
$$+ A^{nn}(A^{n-1n})^{-1}A^{n-1n-1}\left(A^{n-2n-1}\right)^{-1}F^{n-2} + \ldots +$$
$$+ (-1)^{n+1}A^{nn}(A^{n-1n})^{-1}A^{n-1n-1}\left(A^{n-2n-1}\right)^{-1}\ldots$$
$$\ldots A^{33}\left(A^{23}\right)^{-1}A^{22}\left(A^{12}\right)^{-1}F^1. \qquad (18)$$

The vectors C^2, C^3, …, C^n are defined through recurrent relations:

$$C^2 = \left(A^{12}\right)^{-1}\left(F^1 - A^{11}C^1\right),$$

$$C^3 = \left(A^{23}\right)^{-1}\left(F^2 - A^{22}C^2\right),$$

$$\ldots\ldots\ldots\ldots\ldots\ldots$$

$$C^{n-1} = \left(A^{n-2n-1}\right)^{-1}\left(F^{n-2} - A^{n-2n-2}C^{n-2}\right),$$

$$C^n = \left(A^{n-1n}\right)^{-1}\left(F^{n-1} - A^{n-1n-1}C^{n-1}\right). \qquad (19)$$

As we see, in the process of calculations we have to deal not with coefficient matrixes with unknowns of the size $4n \times 4n$, but with the matrixes of the size 4×4. This allows to avoid a great number of computational complexities.

Calculation results

According to the scheme described above certain calculation results were obtained (fig. 2 – 4). For calculations we assumed the following: $n = 3$; $E = 2 \cdot 10^5$ MPa; $\nu = 0{,}25$; $h_1 = 0{,}01$ m; $h_2 = 0{,}02$ m; $h_3 = 0{,}03$ m; $R_1 = 0{,}1$ m; $R_2 = 0{,}2$ m; $R_3 = 0{,}3$ m. The calculation was effectuated for two variants of load $\overline{q}^1 = (1{,}1{,}1)$ MPa (curve 1), $\overline{q}^2 = (1{,}0{,}0)$ MPa (curve 2) (in brackets we indicated the intensity of load on the first, second and third section accordingly).

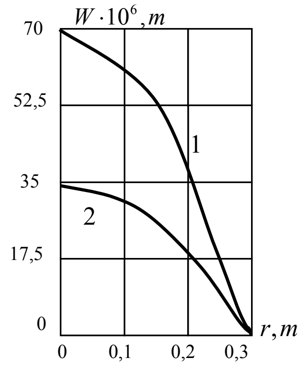

Fig. 2. Regular bendings

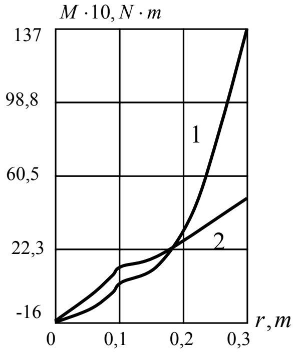

Fig. 3. Bending moments

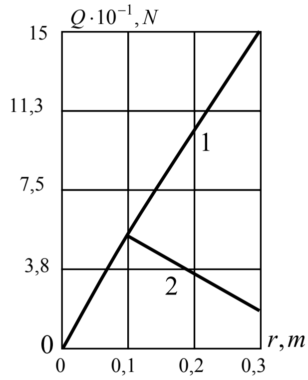

Fig.4. Transverse forces

Literature

1. Биргер М. А., Пановко Я. Г. Прочность, устойчивость, колебания : в 3-х т. – М. : Машиностроение, 1968. – Т. 1. – 832 с.
2. Вайнберг Д. В. Напряженное состояние составных дисков и пластин / Вайнберг Д. В. – К. : Изд-во АН УССР, 1952. – 420 с.
3. Гавеля С. П. Расчет упругого деформирования круглой пластины дискретно-переменной толщины / С. П. Гавеля, С. А. Левчук, О. А. Ищенко. – Запорожье, 1992. – 10 с. – Деп. в УкрИНТЭИ 08.06.92, № 845 – Ук92.
4. Левчук С. А. Матриці Гріна рівнянь та систем еліптичного типу для дослідження статичного деформування складених тіл : дис. ... канд. фіз.-мат. наук : 01.02.04 / Левчук Сергій Анатолійович. – Запоріжжя : ЗДУ, 2002. – 150 с.
5. Левчук С. А. Матриця типу Гріна круглої пластини дискретно-змінної товщини/ С. А. Левчук // Вісник Запорізького державного університету : сер. фізико-математичні науки. – Запоріжжя, 1999. – № 2. – С. 66–69.
6. Левчук С. А. Про деякі способи апроксимації круглих пластин різних профілів/ С. А. Левчук, Ю. О. Сисоєв // Вісник Запорізького державного університету. Сер. Фізико-математичні науки. – Запоріжжя : ЗНУ, 2008. – № 1. – С. 113–117.
7. Крашановська М. О. Про деякі способи розв'язання систем лінійних алгебраїчних рівнянь з квазідіагональними матрицями / М. О. Крашановська, С. А. Левчук, А. А. Хмельницький // Зб. тез доповідей шостої регіональної наукової конференції молодих дослідників присвяченої 90–річчю НАН України «Актуальні проблеми математики та інформатики». – Запоріжжя : ЗНУ, 2008. – С. 30–31.
8. Рибалко О. О. Про розвиток методів розв'язання систем лінійних алгебраїчних рівнянь з квазідіагональними матрицями/ О. О. Рибалко, А. А. Хмельницький, С. А. Левчук // зб. тез доп. Першої Всеукраїнської, восьмої регіональної наукової конференції молодих дослідників присвяченої 80-річчю Запорізького національного університету. – Запоріжжя: ЗНУ, 2010. – С. 36–38.

Одержано 29.04.2014

Левчук С.А. Застосування апарату матриць типу Гріна та матричної алгебри в задачі про статичне деформування круглих пластин дискретно-змінної товщини

Стаття присвячена моделюванню статичного деформування круглих пластин дискретно-змінної товщини. Застосування апарату функцій Гріна та матричної алгебри дозволило побудувати компактний обчислювальний алгоритм розв'язку розглянутої задачі при практично довільній кількості секцій у складеному тілі, яке застосовувалося при моделюванні.

Ключові слова: *кругла пластина дискретно-змінної товщини, крайова та складена задача, складена конструкція, матриця типу Гріна , матрична алгебра.*

Левчук С.А. Применение аппарата матриц типа Грина и матричной алгебры в задаче про статическое деформирование круглых пластин дискретно-переменной толщины

Статья посвящена моделированию статического деформирования круглых пластин дискретно-переменной толщины. Применение аппарата функций Грина и матричной алгебры позволило построить компактный вычислительный алгоритм решения рассмотренной задачи при практически произвольном количестве секций в составном теле, которое применялось при моделировании.

Ключевые слова: *круглая пластина дискретно-переменной толщины, гранично-составная задача, составная конструкция, матрица типа Грина, матричная алгебра.*